PSCB57 - PROF. HANNO REIN

BAYES'S THEOREM

PLAN:

1. Probability
2. Conditional probability
3. Bayes's theorem
4. Bayesian statistics

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1. Probability
2. Conditional probability
3. Bayes’s theorem
4. Bayesian statistics
5. Example

PROBABILITY

- A number between 0 and 1
- Represents a degree of belief in a fact or prediction
- 1 means outcome is certainly true
- 0 represents certainty that fact is false

CONDITIONAL PROBABILITY

- A probability based on some background knowledge

CONDITIONAL PROBABILITY

Example: What is the probability of me having a heart attack in the next year?

According to the CDC 785,000 Americans have a heart attack. There are 311 million Americans.

Thus a random person has a heart attach with a probability of 0.3%.

CONDITIONAL PROBABILITY

Obviously not completely random. Age, blood pressure, smoking, etc play a role.

CONDITIONAL PROBABILITY

Notation:

$$p(A|B)$$

Probability of A given that B is true. A is the prediction. B is a set of conditions.

CONJOINT PROBABILITY

Fancy way of saying: probability that two things are true.

$$p(A \text{ and } B)$$

CONJOINT PROBABILITY

Example 1: Tossing two coins

$$p(A \text{ and } B) = p(A)\, p(B)$$

Probability that first coin lands face up:

$$p(A) = 0.5$$

Probability that second coin lands face up:

$$p(B) = 0.5$$

Works only because coin tosses are independent!

CONJOINT PROBABILITY

Example 2:
A means it rains today. B means it rains tomorrow.

If we know it rained today, then it's more likely that it rains tomorrow.

$$p(B|A) > p(B)$$

CONJOINT PROBABILITY

In general, the probability of a conjunction:

$$p(A \text{ and } B) = p(A)\, p(B|A)$$

So if the probability that it rains on any given day is 0.5. Then the probability of it raining on two consecutive days is greater than 0.25.

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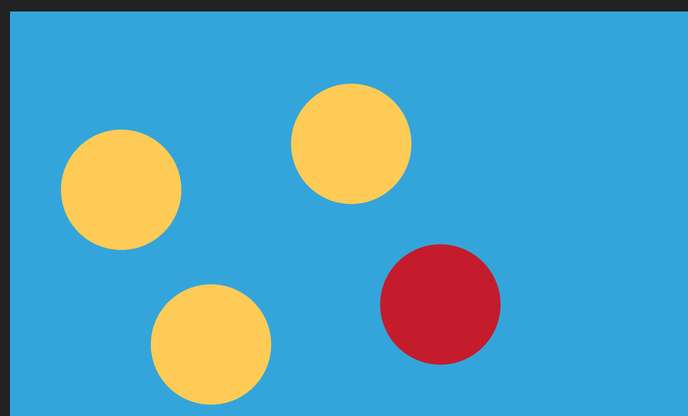
COOKIE PROBLEM

Two bowls of cookies.



30 vanilla

10 chocolate

20 vanilla

20 chocolate

Choose a bowl randomly, then choose a cookie randomly. It’s vanilla. What is the probability it came from bowl 1?

COOKIE PROBLEM

This is a hard question! Asking a different question is easy:

What is the probability of a vanilla cookie in bowl 1?

$$p(\text{vanilla}|\text{bowl 1}) = 3/4$$

$$p(\text{bowl 1}|\text{vanilla}) = ?$$

BAYES'S THEOREM

Idea: use Bayes's theorem to calculate

$$p(\text{bowl } 1|\text{vanilla}) = ?$$

DERIVATION OF BAYES'S THEOREM

Note that:

$$p(A \text{ and } B) = p(B \text{ and } A)$$

Writing down the conjoined probabilities:

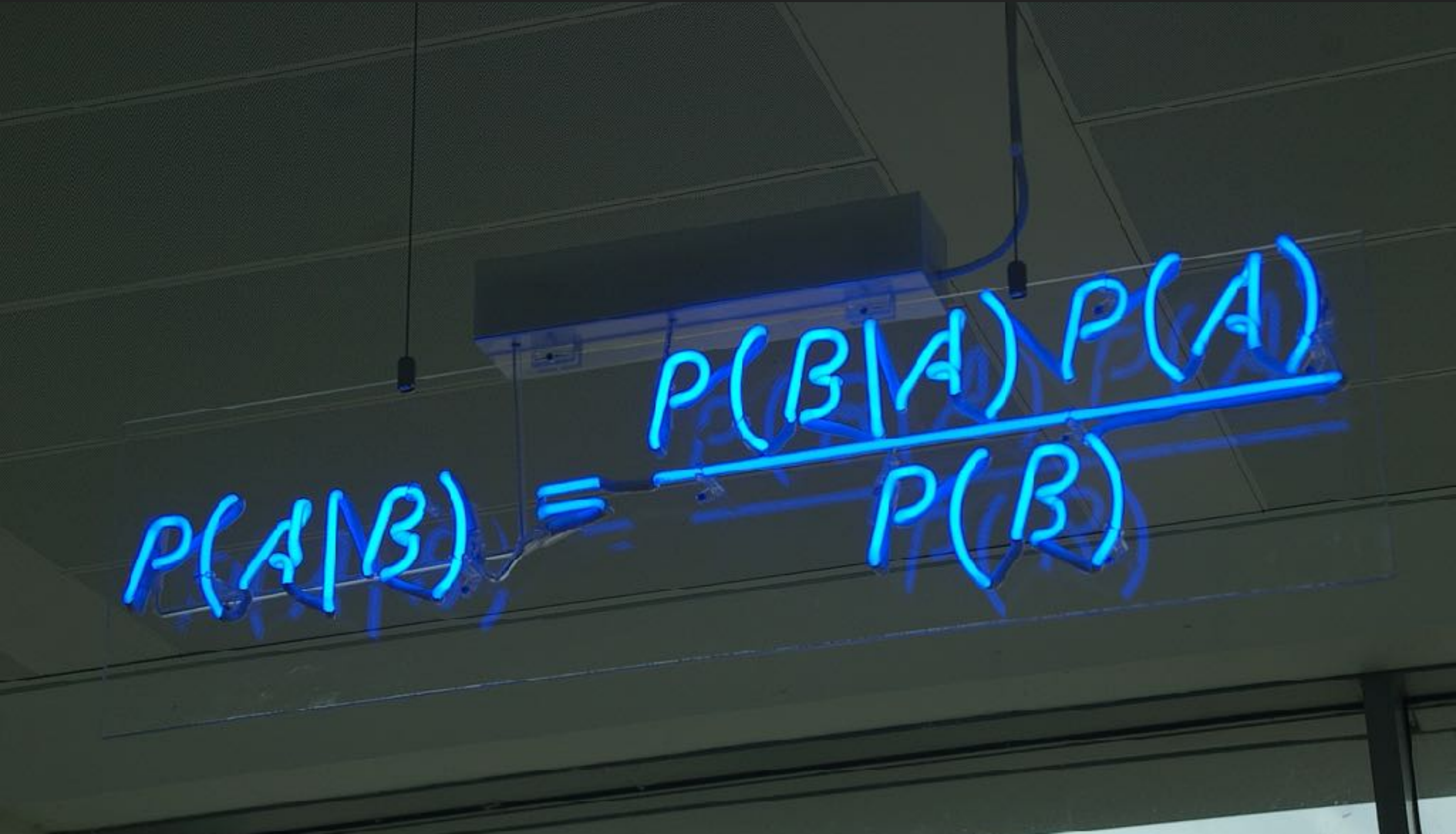
$$p(A \text{ and } B) = p(A)\, p(B|A)$$

$$p(B \text{ and } A) = p(B)\, p(A|B)$$

This gives us:

$$p(A|B) = \frac{p(A)\, p(B|A)}{p(B)}$$

BAYES'S THEOREM



COOKIE PROBLEM REVISITED

Let's call B_1 the hypothesis that the cookie came from bowl 1 and V for vanilla. Bayes's theorem gives us:

$$p(B_1|V) = \frac{p(B_1)p(V|B1)}{p(V)}$$

COOKIE PROBLEM REVISITED

$$p(B_1|V) = \frac{p(B_1)p(V|B1)}{p(V)}$$

$$p(B_1) = \frac{1}{2}$$

$$p(V|B_1) = \frac{3}{4}$$

$$p(V) = \frac{5}{8}$$

$$p(B_1|V) = \frac{\frac{1}{2}\ \frac{3}{4}}{\frac{5}{8}} = \frac{3}{5}$$

DIACHRONIC INTERPRETATION

One way of thinking about Bayes's theorem. Suppose we have a hypothesis H and some data D.

$$p(H|D) = \frac{p(H)\ p(D|H)}{p(D)}$$

These terms now have names and can be interpreted as follows.

DIACHRONIC INTERPRETATION

$$p(H|D) = \frac{p(H)\, p(D|H)}{p(D)}$$

$p(H)$	Prior
$p(H\|D)$	Posterior
$p(D\|H)$	Likelihood
$p(D)$	Normalization constant

DIACHRONIC INTERPRETATION

- Prior is subjective. But that is ok as long as you can write it down.
- Likelihood is easy to compute.
- Normalization constant is tricky. Often this is a high dimensional integral. Monte Carlo methods!

MONTY HALL PROBLEM