BEFORE WE START…

QUIZ

ASSIGNMENT 2 IS OUT, DUE OCTOBER 6TH

FORM GROUPS FOR GEOTAB UNTIL NEXT WEEK

PSCB57 - PROF. HANNO REIN

MATRICIES

LINEAR SYSTEM OF EQUATIONS

$$y_0 = a_{00} \cdot x_0 + a_{01} \cdot x_1$$

$$y_0 = a_{10} \cdot x_0 + a_{11} \cdot x_1$$

LINEAR SYSTEM OF EQUATIONS

$$y_0 = a_{00} \cdot x_0 + a_{01} \cdot x_1$$

$$y_0 = a_{10} \cdot x_0 + a_{11} \cdot x_1$$

LINEAR SYSTEM OF EQUATIONS WITH MATRICES

$$\begin{pmatrix} y_0 \\ y_1 \end{pmatrix} = \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} \cdot \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}$$

MATRICES IN PYTHON

$$\begin{pmatrix} y_0 \\ y_1 \end{pmatrix} = \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} \cdot \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}$$

Lists:

[y0, y1]
[x0,x1]

and a list of lists:
[[a00, a01], [a10, a11]]

LINEAR SYSTEM OF EQUATIONS

$$\begin{pmatrix} y_0 \\ y_1 \end{pmatrix} = \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} \cdot \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}$$

Often, we want to solve for x!

PSCB57 - PROF. HANNO REIN

ALGORITHMIC COMPLEXITY

FIBONACCI NUMBER EXAMPLE

```
def g2(x):
    if x==0:
        return 0
    if x==1:
        return 1
    return g2(x-1)+g2(x-2)
```

$$O(2^N)$$

MATH

FIBONACCI NUMBERS

$$F_n = F_{n-1} + F_{n-2}$$
$$F_{n-1} = F_{n-1}$$

FIBONACCI NUMBERS

$$F_n = 1 \cdot F_{n-1} + 1 \cdot F_{n-2}$$

$$F_{n-1} = 1 \cdot F_{n-1} + 0 \cdot F_{n-2}$$

FIBONACCI NUMBERS

$$\begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} F_{n-1} \\ F_{n-2} \end{pmatrix}$$

FIBONACCI NUMBERS

$$\vec{F}_{n+1} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \cdot \vec{F}_n$$

$$\vec{F}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

FIBONACCI NUMBERS

$$\vec{F}_n = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n-1} \cdot \vec{F}_1$$

$$\vec{F}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

TRY TO FIND THE EIGENVALUES AND EIGENVECTORS OF

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

EIGENVALUES AND EIGENVECTORS

$$\lambda_1 = \frac{1+\sqrt{5}}{2} \quad \text{with} \quad \vec{x}_1 = \begin{pmatrix} \frac{1}{2}(1+\sqrt{5}) \\ 1 \end{pmatrix}$$

$$\lambda_2 = \frac{1-\sqrt{5}}{2} \quad \text{with} \quad \vec{x}_2 = \begin{pmatrix} \frac{1}{2}(1-\sqrt{5}) \\ 1 \end{pmatrix}$$

EIGENVALUES AND EIGENVECTORS

$$\lambda_1 = \frac{1+\sqrt{5}}{2} \quad \text{with} \quad \vec{x}_1 = \begin{pmatrix} \frac{1}{2}(1+\sqrt{5}) \\ 1 \end{pmatrix}$$

$$\lambda_2 = \frac{1-\sqrt{5}}{2} \quad \text{with} \quad \vec{x}_2 = \begin{pmatrix} \frac{1}{2}(1-\sqrt{5}) \\ 1 \end{pmatrix}$$

$$\vec{F}_n = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n-1} \cdot \vec{F}_1$$

$$\vec{F}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\vec{F}_1 = \frac{1}{\sqrt{5}} \left(\vec{x}_1 - \vec{x}_2\right)$$

EIGENVALUES AND EIGENVECTORS

$$\lambda_1 = \frac{1+\sqrt{5}}{2} \quad \text{with} \qquad \vec{x}_1 = \begin{pmatrix} \frac{1}{2}(1+\sqrt{5}) \\ 1 \end{pmatrix}$$

$$\lambda_2 = \frac{1-\sqrt{5}}{2} \quad \text{with} \qquad \vec{x}_2 = \begin{pmatrix} \frac{1}{2}(1-\sqrt{5}) \\ 1 \end{pmatrix}$$

$$\vec{F}_n = \frac{1}{\sqrt{5}} \left(\lambda_1^{n-1}\vec{x}_1 - \lambda_2^{n-1}\vec{x}_2\right)$$

EIGENVALUES AND EIGENVECTORS

$$\lambda_1 = \frac{1+\sqrt{5}}{2} \quad \text{with} \quad \vec{x}_1 = \begin{pmatrix} \frac{1}{2}(1+\sqrt{5}) \\ 1 \end{pmatrix}$$

$$\lambda_2 = \frac{1-\sqrt{5}}{2} \quad \text{with} \quad \vec{x}_2 = \begin{pmatrix} \frac{1}{2}(1-\sqrt{5}) \\ 1 \end{pmatrix}$$

$$\begin{aligned} F_n &= \frac{1}{\sqrt{5}} \left(\lambda_1^{n-1} \frac{1}{2}(1+\sqrt{5}) - \lambda_2^{n-1} \frac{1}{2}(1-\sqrt{5}) \right) \\ &= \frac{1}{\sqrt{5}} \left(\left(\frac{1}{2}(1+\sqrt{5}) \right)^n - \left(\frac{1}{2}(1-\sqrt{5}) \right)^n \right) \end{aligned}$$

ALGORITHM TO CALCULATE FIBONACCI NUMBERS

$$F_n = \frac{1}{\sqrt{5}}\left(\lambda_1^{n-1}\frac{1}{2}(1+\sqrt{5}) - \lambda_2^{n-1}\frac{1}{2}(1-\sqrt{5})\right)$$
$$= \frac{1}{\sqrt{5}}\left(\left(\frac{1}{2}(1+\sqrt{5})\right)^n - \left(\frac{1}{2}(1-\sqrt{5})\right)^n\right)$$

$$O(1)$$

YET ANOTHER ALGORITHM

Proof by induction:

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$$

For any matrix A and $n \geq 0$:

$$A^{2n} = A^n A^n$$

This gives the relations:

$$F_{2n-1} = F_n^2 + F_{n-1}^2$$
$$F_{2n} = (F_{n-1} + F_{n+1})F_n = (2F_{n-1} + F_n)F_n$$

WHY IS THIS 3RD ALGORITHM USEFUL

odd numbers

$$F_{2n-1} = F_n^2 + F_{n-1}^2$$

$$F_{2n} = (F_{n-1} + F_{n+1})F_n = (2F_{n-1} + F_n)F_n$$

even numbers

WRONG! NOT TRUE!

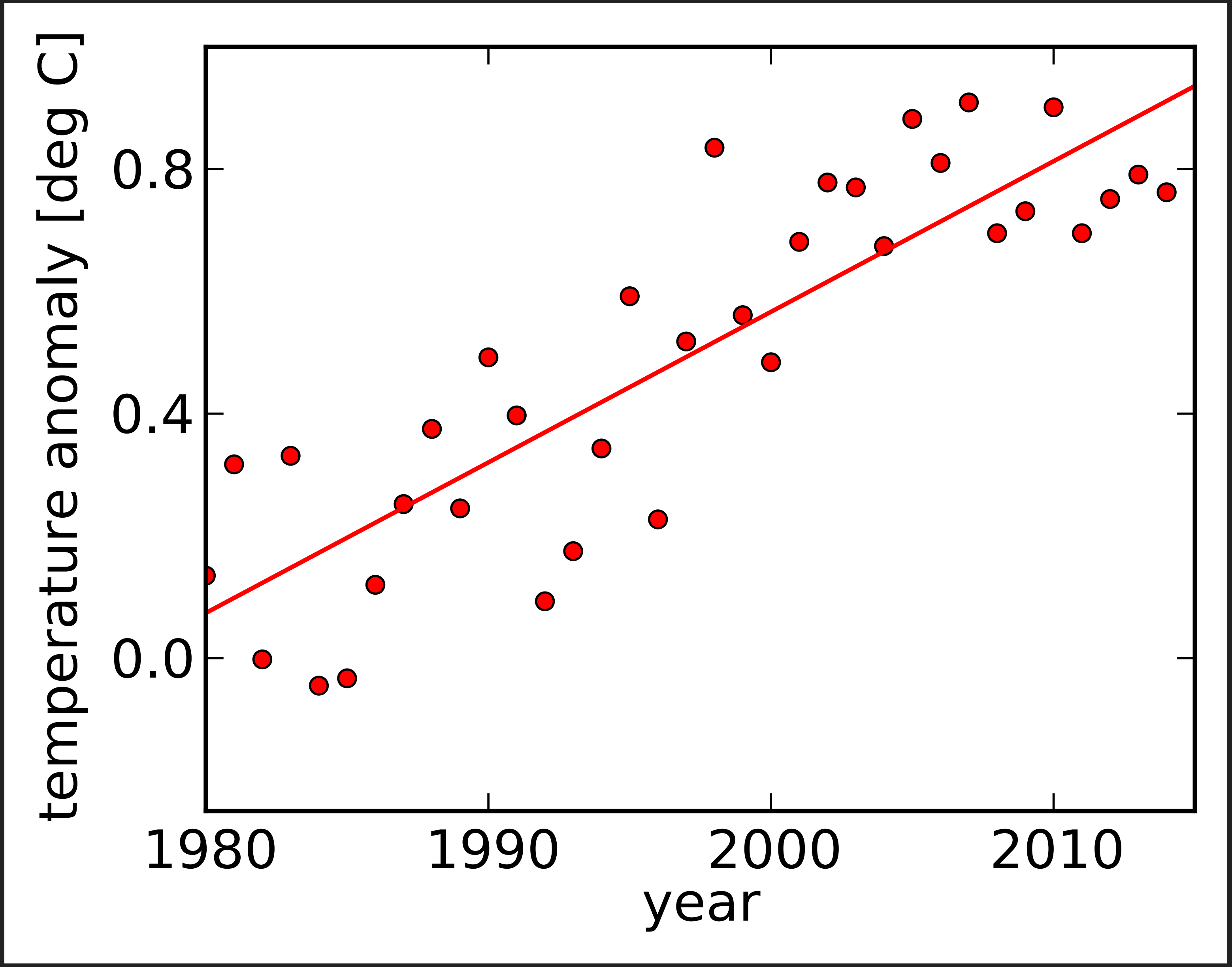
Still recursive, but … $O(\log(N))$

… and can be done in integers!

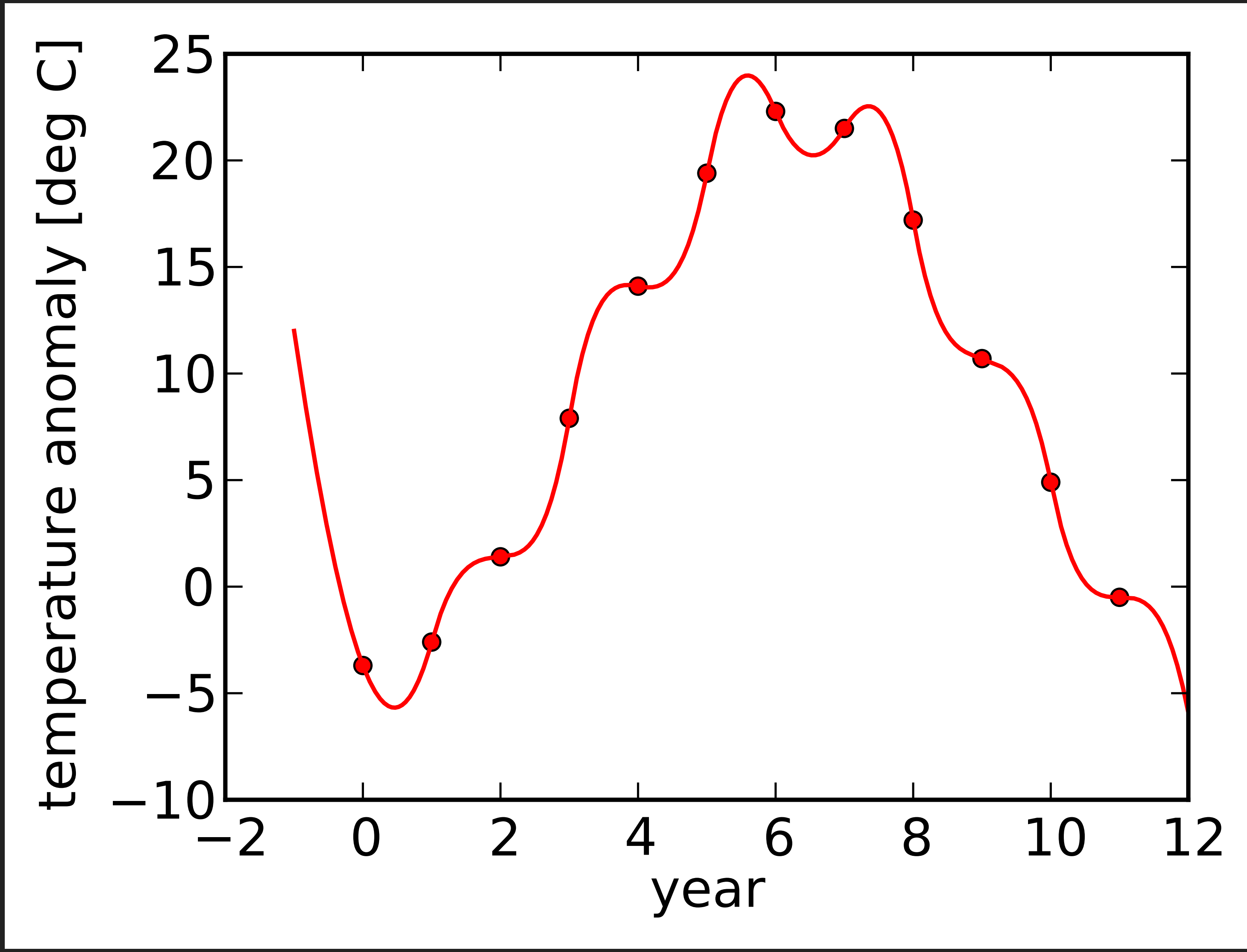
PSCB57 - PROF. HANNO REIN

LINEAR LEAST SQUARE FIT

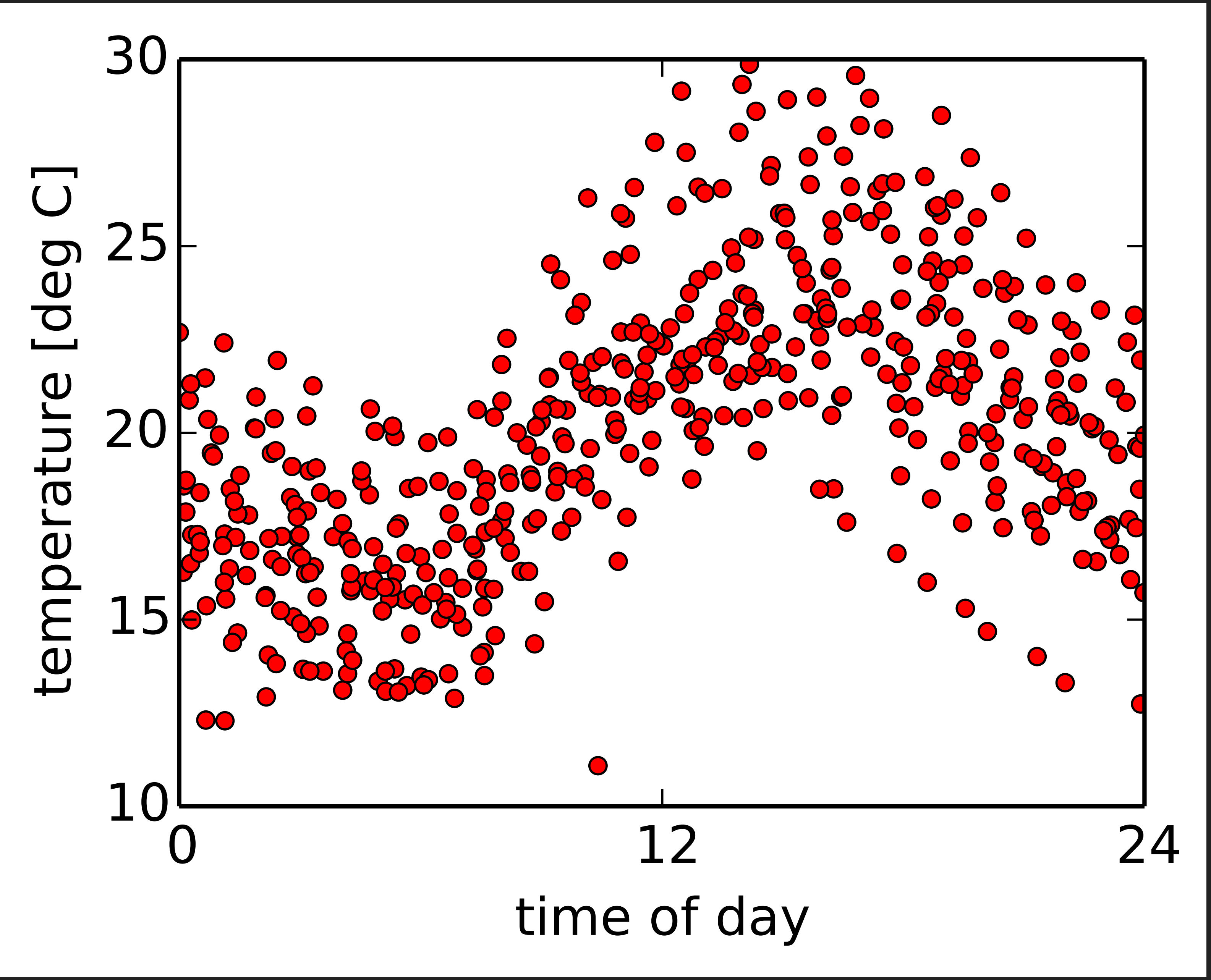
LINEAR LEAST SQUARE FIT

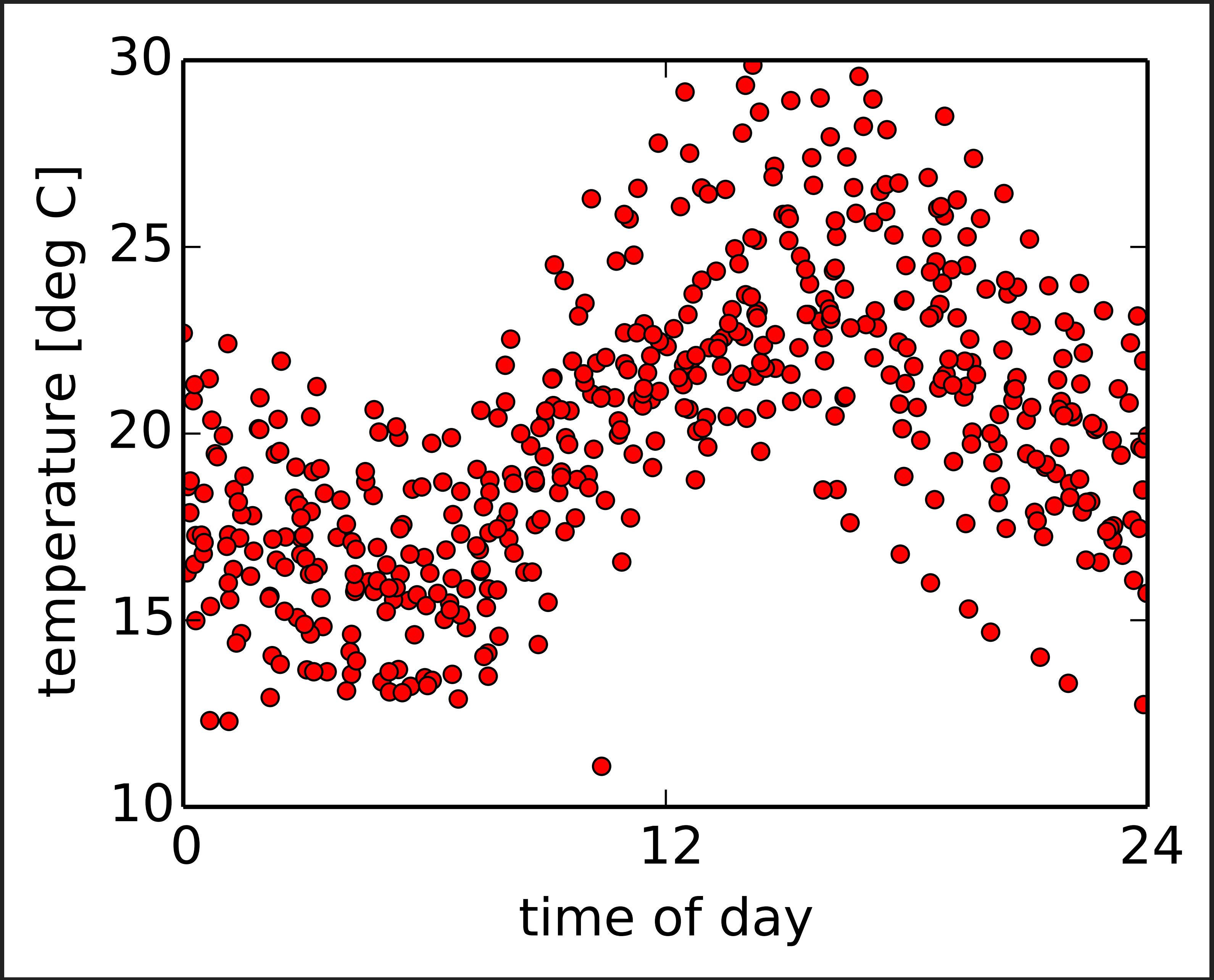


LINEAR LEAST SQUARE FIT



LINEAR LEAST SQUARE FIT



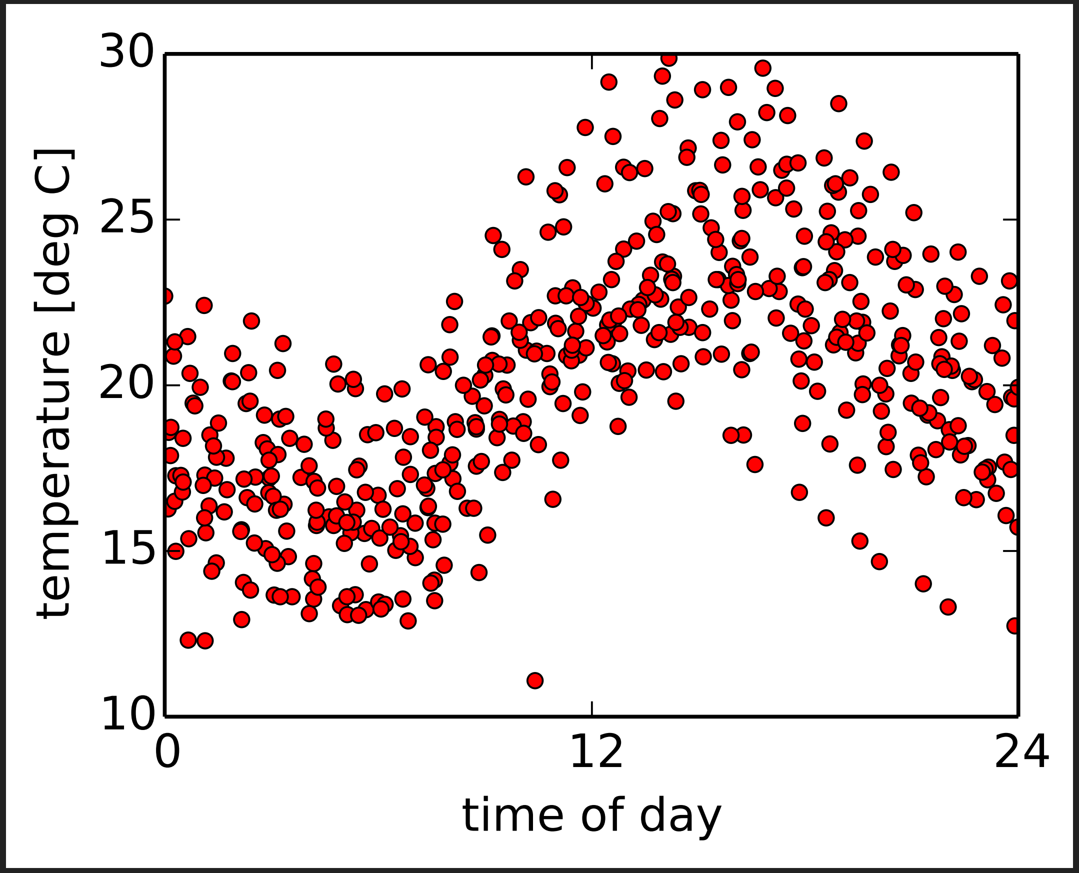


$$f(t) = a_0 + a_1 \sin\left(\frac{t}{24}2\pi\right) + a_2 \cos\left(\frac{t}{24}2\pi\right)$$

LINEARITY

$$f(t) = a_0 + a_1 \sin\left(\frac{t}{24}2\pi\right) + a_2 \cos\left(\frac{t}{24}2\pi\right)$$

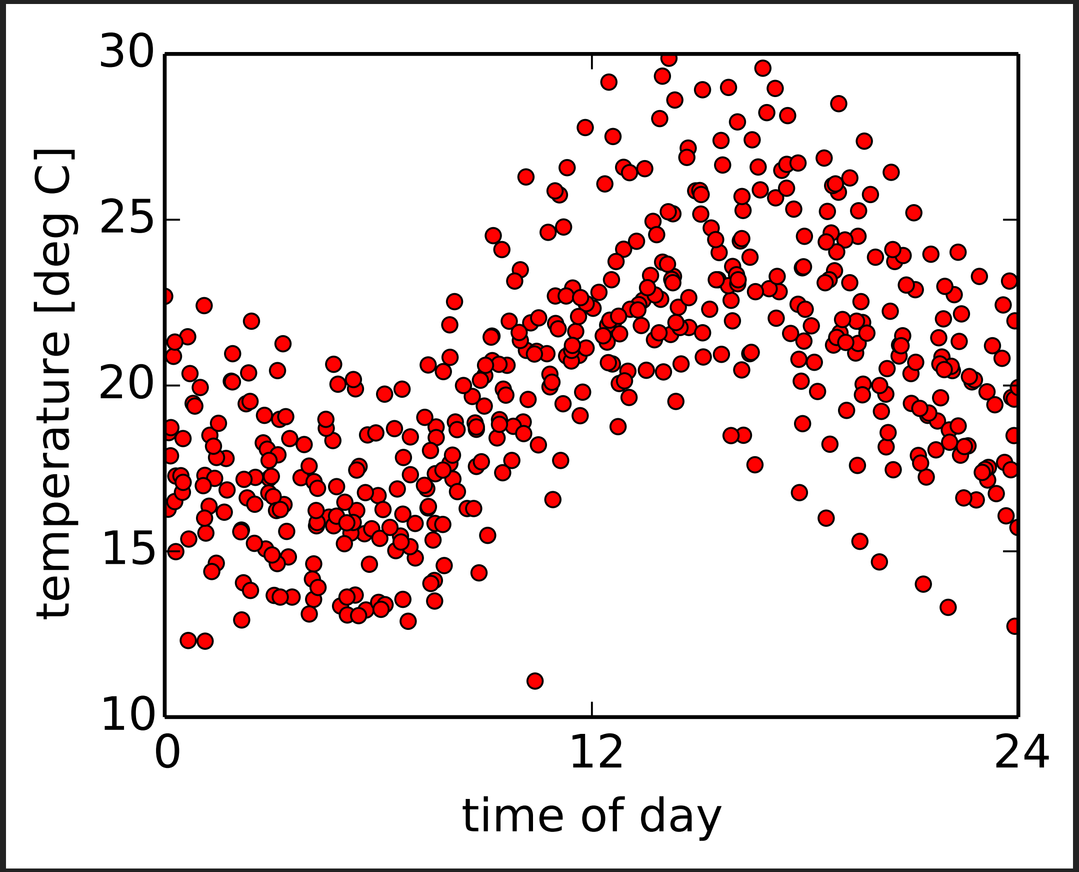
$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$



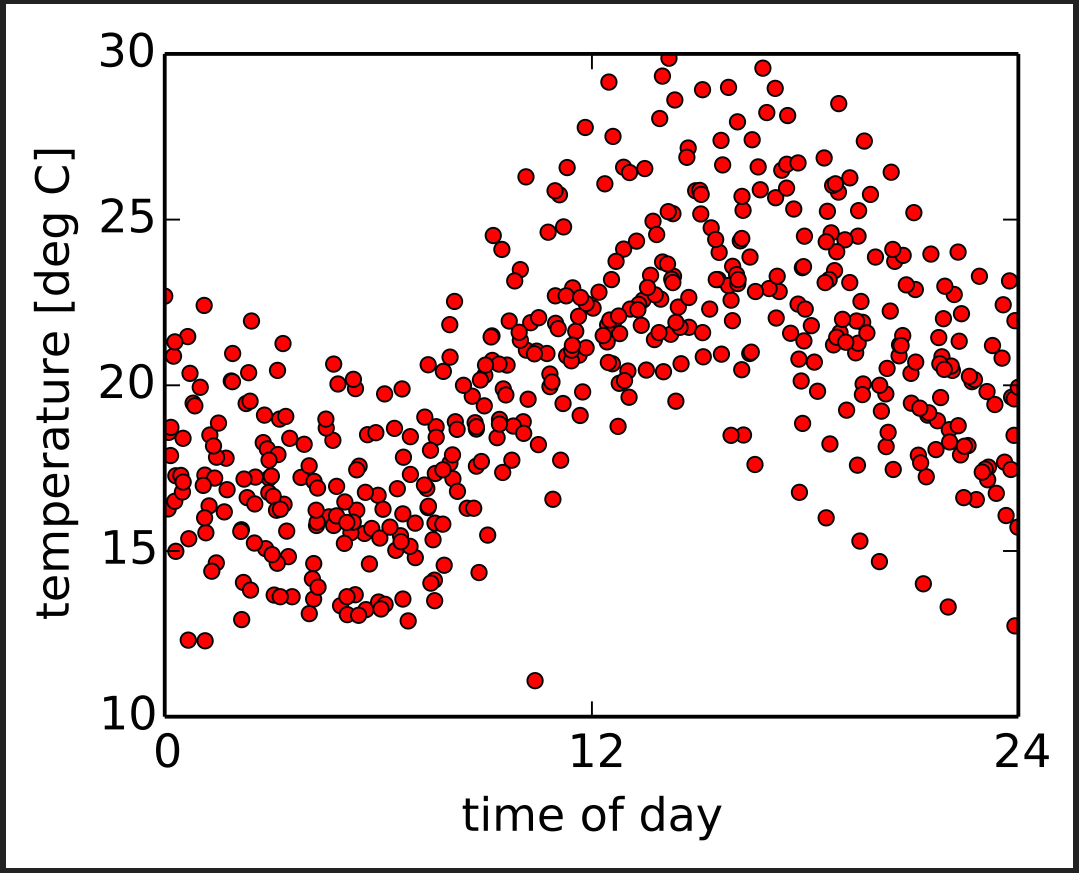
$$y_0 = f(x_0) = a_0 + a_1 \sin\left(\frac{x_0}{24} 2\pi\right) + a_2 \cos\left(\frac{x_0}{24} 2\pi\right)$$

$$y_1 = f(x_1) = a_0 + a_1 \sin\left(\frac{x_1}{24} 2\pi\right) + a_2 \cos\left(\frac{x_1}{24} 2\pi\right)$$

$$y_2 = f(x_2) = a_0 + a_1 \sin\left(\frac{x_2}{24} 2\pi\right) + a_2 \cos\left(\frac{x_2}{24} 2\pi\right)$$



$$\begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & \sin\left(\frac{x_0}{24}2\pi\right) & \cos\left(\frac{x_0}{24}2\pi\right) \\ 1 & \sin\left(\frac{x_1}{24}2\pi\right) & \cos\left(\frac{x_1}{24}2\pi\right) \\ 1 & \sin\left(\frac{x_2}{24}2\pi\right) & \cos\left(\frac{x_2}{24}2\pi\right) \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}$$



$$\begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_{N-1} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & \sin\left(\frac{x_0}{24}2\pi\right) & \cos\left(\frac{x_0}{24}2\pi\right) \\ 1 & \sin\left(\frac{x_1}{24}2\pi\right) & \cos\left(\frac{x_1}{24}2\pi\right) \\ & \vdots & \\ 1 & \sin\left(\frac{x_{N-1}}{24}2\pi\right) & \cos\left(\frac{x_{N-1}}{24}2\pi\right) \end{pmatrix}}_{\equiv C} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} e_0 \\ e_1 \\ \vdots \\ e_{N-1} \end{pmatrix}$$

LEAST SQUARE

minimize: $$S = \sum_{i=0}^{N-1} e_i^2$$

minimize: $$S = \sum_{i=0}^{N-1} e_i^2$$

$$0 = \frac{\partial S}{\partial a_j} = 2\sum_{i=0}^{N-1} e_i \frac{\partial e_i}{\partial a_j} \qquad \forall j$$

$$= 2\sum_{i=0}^{N-1}\left[y_i - \sum_{k=0}^{m-1}\left(a_k C_{ik}\right)\right] C_{ij}$$

minimize: $$S = \sum_{i=0}^{N-1} e_i^2$$

$$0 = \frac{\partial S}{\partial a_j} = 2\sum_{i=0}^{N-1} e_i \frac{\partial e_i}{\partial a_j} \qquad \forall j$$

$$= 2\sum_{i=0}^{N-1}\left[y_i - \sum_{k=0}^{m-1}(a_k C_{ik})\right] C_{ij}$$

rearranging gives:

$$\sum_{i=0}^{N-1} C_{ij} y_i = \sum_{i=0}^{N-1}\sum_{k=0}^{m-1} C_{ik} C_{ij} a_k$$

$$\sum_{i=0}^{N-1} C_{ij} y_i = \sum_{i=0}^{N-1} \sum_{k=0}^{m-1} C_{ik} C_{ij} a_k$$

In matrix notation:

$$\underbrace{C^T \cdot y}_{b} = \underbrace{\left(C^T C\right)}_{A} \cdot a$$

SUMMARY

- Linear Least Square Fit
- Can be used to fit functions to data
- Dependence on parameters must be linear
- Functions themselves do not need to be linear
- Leads to a linear system of equations

$$\underbrace{C^T \cdot y}_{b} = \underbrace{\left(C^T C\right)}_{A} \cdot a$$

PSCB57 - PROF. HANNO REIN

HOW TO SOLVE A LINEAR SYSTEM OF EQUATIONS

See lecture notes on the LU decomposition.