Name: ______________________________ Student number: ______________________________

Introduction to Scientific Computing
PSCB57, Final Exam

Professor Hanno Rein
University of Toronto, Scarborough
Wednesday, December 7th, 2016, 2pm-4pm

	Points	Max Points
Question 1		6
Question 2		6
Question 3		8
Question 4		8
Question 5		5
Question 6		6
Question 7		9
Question 8		5
Question 9		5
Total		58

- No calculators, aid sheets, books or other notes are allowed.
- All electronic devices must be stored together with your belongings at the back of the room.
- Write your answers on the question sheet. If you need more paper raise your hand.
- The length of the white space for each question gives you an idea of the expected length and complexity of the answer.
- If you leave the room, please do not disturb others. You can not leave in the last 10 minutes.
- The University of Toronto's Code of Behaviour on Academic Matters applies to all University of Toronto Scarborough students. The Code prohibits all forms of academic dishonesty including, but not limited to, cheating, plagiarism, and the use of unauthorized aids. Students violating the Code may be subject to penalties up to and including suspension or expulsion from the University.

Question 1 **6 Points**

Mark if the following numbers can be represented as an IEEE754 double floating point number exactly, approximately, or not at all.

	exactly	approximately	not at all
`0.375`	☐	☐	☐
`3.1`	☐	☐	☐
π	☐	☐	☐
`32.0`	☐	☐	☐
`-2e+1`	☐	☐	☐
`0.125`	☐	☐	☐
`1e-300`	☐	☐	☐
`-1e-1`	☐	☐	☐
`1e+399`	☐	☐	☐

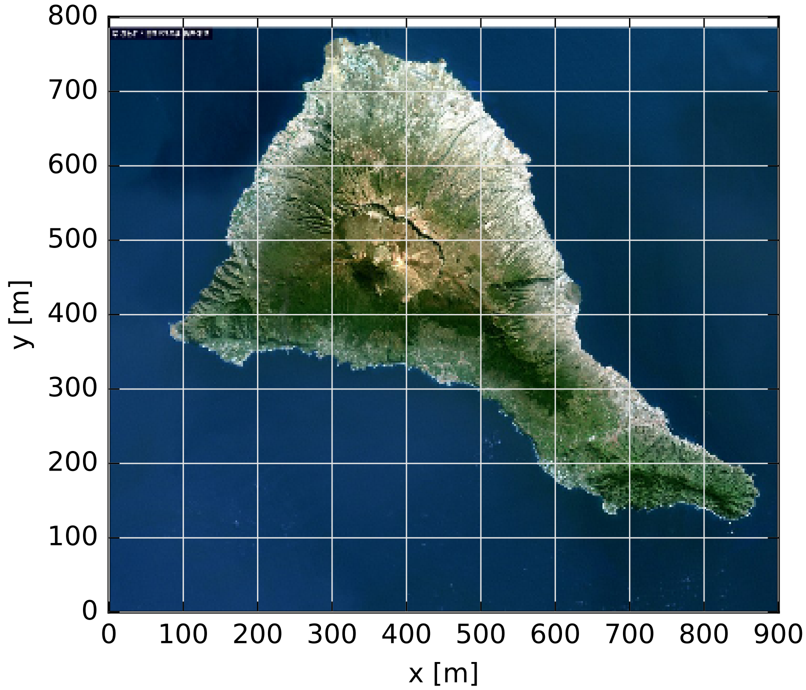
Calculate the following expressions in IEEE754 double floating point precision.

`(1e-100 /1e+100 ) + 1.5 =` ____________________

`((1e-8 + 1e+10) - 2e-8) - 1e10 =` ____________________

Question 2

6 Points



A random number generator has been used to come up with the following coordinates in the interval $x \in [0 : 900]$ and $y \in [0 : 800]$.

x	y	x	y	x	y	x	y
276	671	386	19	135	303	76	718
718	594	771	654	738	304	346	564
800	82	627	235	106	606	320	714

Use these coordinates and the Monte Carlo method to estimate the area of the island in m^2. Quote the exact outcome of your calculation (no rounding):

What do you have to assume about the random numbers for this method to work?

Question 3 **8 Points**

Monty Hall, the original host of the game show *Let's Make a Deal*, shows you **four** closed doors. A prize is placed behind one of the doors at random. The objective of the game is to guess which door contains the prize.

- You pick a door, which we will call door A. We'll call the other doors B, C and D.
- Before opening the door you chose, Monty Hall increases the suspense by opening either door B, C or D. If the prize is behind A, he opens one of the doors B, C or D at random. If the prize is behind B, C or D, he opens one of the two doors that do not contain the prize. Between the two doors he chooses again at random.
- In our case Monty Hall opens the door that we call B.
- Then Monty Hall offers you the option to stick with your original choice or to switch to one of the other unopened doors.

To calculate the best strategy to win the prize, we will use Bayes' Theorem:

$$p(H|D) = \frac{p(H)\, p(D|H)}{p(D)}$$

We work in the diachronic interpretation. In our case the data D corresponds to the event *"Monty Hall opens door B and there is no prize behind it"*. There are four different hypotheses H, each corresponds to *"the prize is behind door X"*.
Fill out the following table to calculate the posterior probability of the different hypotheses.

H	Prior $p(H)$	Likelihood $p(D\|H)$	$p(H)\, p(D\|H)$	Posterior $p(H\|D)$
A				
B				
C				
D				

What is the best strategy to win the prize?

Question 4

8 Points

Three terrestrial planets in our Solar System have the following masses and radii.

Planet	Radius ($0.5R_e$)	Mass (M_e)
Venus	2	0.8
Earth	2	1
Mars	1	0.1

Use a linear least square fit to fit the function

$$M(r) = a_0 \cdot r^3$$

Here, a_0 is the free parameter to fit for, r is the radius, and M is the mass. Writing down the answer as a fraction is ok.

Suppose that you want to fit one of the following functions instead of the function $M(r)$ from above. Can you use a linear least square fit with these functions? If not, can you rewrite them so that you can use a standard linear least square fit?

$M_1(r) = a_0 \cos(\pi \cdot r + a_1)$

$M_2(r) = a_0 \left[Y_3(r)\right]^{5.3} + a_1 \cdot \log\left(\sqrt{r}\right)$, where Y_3 is a Bessel function of second kind.

Describe in words, what is the difference between a fit and an interpolation?

Question 5

5 Points

Find the Lagrange polynomial that interpolates the following (x, y) points. Do not simplify the polynomial.

$$(-12, \alpha)$$
$$(\sqrt{2}, 4^{-1/3})$$
$$(\pi, 1000)$$
$$(c, 42)$$

Describe some of the difference between cubic spline and Lagrange interpolation.

Question 6

6 Points

```
def g1(x):
    if x==0:
        return 0
    if x==1:
        return 1
    if x%2==0:
        gx = g1(x/2)
        gx1 = g1(x/2-1)
        return (2*gx1+gx)*gx
    if x%2==1:
        gx = g1((x+1)/2)
        gx1 = g1((x+1)/2-1)
        return gx*gx+gx1*gx1

def g2(x):
    if x==0:
        return 0
    if x==1:
        return 1
    return g2(x-1)+g2(x-2)

def g3(x):
    sqrt5 = 2.23606797749978 98
    return ((1+sqrt5)**x-(1-sqrt5)**x)/(2**x*sqrt5)
```

What are the above python functions calculating?

What is the algorithmic complexity of each function?

g1(x):

g2(x):

g3(x):

If you need an exact result, which function(s) can you use?

If you want a fast approximate result for large x, which function(s) would you use?

Question 7 **9 Points**

You are given the differential equation

$$\frac{\partial^2 x(t)}{\partial t^2} = 2 \cdot x$$

with initial conditions

$$\begin{aligned} x(0) &= 0 \\ \left.\frac{\partial x(t)}{\partial t}\right|_{t=0} &= 1. \end{aligned}$$

Rewrite the second order differential equation as a coupled system of two first order differential equations.

Perform two Euler steps with a time-step of $h = 0.5$.

What is your approximation for the value of x and its first derivative at $t = 1$?

$x(1) \approx$ ________________________________ $\left.\frac{\partial x(t)}{\partial t}\right|_{t=1} \approx$ ________________________________

This is a bad approximation of the true solution. What could you do to improve the accuracy?

Question 8

5 Points

State the intermediate value theorem.

Explain why the intermediate value theorem is important for the bisection method. What about Newton's method?

Question 9

5 Points

We want to calculate the integral

$$\int_0^2 (3x^2 - 2x + 1)\mathrm{d}x.$$

What is the analytic solution?

Use the trapezoidal rule with 1 slice (1 trapezoid) to estimate the integral.

Use the trapezoidal rule with 2 slices (2 trapezoid) to estimate the integral.