Last name: ______________________ First name: ______________________

Student number: ______________________

Mechanics: From Oscillations to Chaos, PHYB54, Final Exam

Professor Hanno Rein
University of Toronto, Scarborough

Tuesday, April 10th 2019, 9am - 11am

	Points	Max Points
Question 1		3
Question 2		8
Question 3		5
Question 4		3
Question 5		2
Question 6		8
Question 7		4
Total Points		33

- No aid sheets, books, or other notes are allowed.
- You may use a non-programmable calculator and a ruler.
- Write your answers directly on the question sheet. If you need more paper raise your hand.
- The University of Toronto's Code of Behaviour on Academic Matters applies to all University of Toronto Scarborough students. The Code prohibits all forms of academic dishonesty including, but not limited to, cheating, plagiarism, and the use of unauthorized aids. Students violating the Code may be subject to penalties up to and including suspension or expulsion from the University.

Question 1 **3 Points**

Prove that if an arbitrary Lagrangian $\mathcal{L}$ for a system with N particles is translationally invariant, i.e.

$$\mathcal{L}(\mathbf{r}_1, \dot{\mathbf{r}}_1, \mathbf{r}_2, \dot{\mathbf{r}}_2, \ldots, \mathbf{r}_N, \dot{\mathbf{r}}_N) = \mathcal{L}(\mathbf{r}_1 + \epsilon, \dot{\mathbf{r}}_1, \mathbf{r}_2 + \epsilon, \dot{\mathbf{r}}_2, \ldots, \mathbf{r}_N + \epsilon, \dot{\mathbf{r}}_N) \qquad \forall \epsilon,$$

then the total momentum is conserved.

Question 2 **8 Points**

A mass m can move horizontally in one dimension along the x-axis. It is attached to walls on each side via two massless springs with spring constants k_1 and k_2. The equilibrium position is at $x = 0$. The mass is also subject to linear air drag with coefficient b. Write down the total force acting on the mass and use Newton's law to derive the equation of motion!

Convert the equation of motion to a set of first order differential equations! To do that, use the variable $\mathbf{y}$ defined as

$$\mathbf{y} \equiv \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \equiv \begin{pmatrix} x \\ \dot{x} \end{pmatrix} .$$

Hint: Your answer should only include $\mathbf{y}$ *and its components* y_1 *and* y_2*, but neither* x*, nor* $\dot{x}$*.*

You want to use scipy's `odeint` integrator. Write a python function which implements the right hand side of the differential equation from above! *Try to be as accurate as possible, however, small syntax errors will not be penalized.*

```
def rhs(y,t,m,k1,k2,b):
```

Give expressions for:

(a) the natural frequency of the undamped oscillator

(b) the frequency of the damped oscillator

(c) the value of ω at which the response is maximal given an additional external sinusoidal driving force $F_{\text{ext}}(t)$ with amplitude $m\, f_0$ and frequency w

Ignoring any transients, we now look at the long term evolution of the forced system. Depending on ω, the system can be in various different regimes. Name those regimes and comment on the amplitude and phase of the solution in them!

Regime 1:

Regime 2:

Regime 3:

Question 3 **5 Points**

Consider a basketball thrown during a Toronto Raptors game. Ignoring gravity, the only forces acting on it are the linear and quadratic drag forces given by

$$\begin{aligned} f_{\text{lin}} &= \beta\, D\, v \\ f_{\text{quad}} &= \gamma\, D^2\, v^2, \end{aligned}$$

where D is the diameter of the ball, $\beta = 1.6 \cdot 10^{-4} N\, s/m^2$ and $\gamma = 0.25 N\, s^2/m^4$. Estimate which drag force dominates!

Taking only the dominant drag force into account, write down the differential equation for $v(t)$, then derive the general solution! Once again, ignore gravity.

Prove that the solutions do not satisfy the superposition principle!

Discuss the superposition principle in the context of a chaotic system!

Question 4 **3 Points**

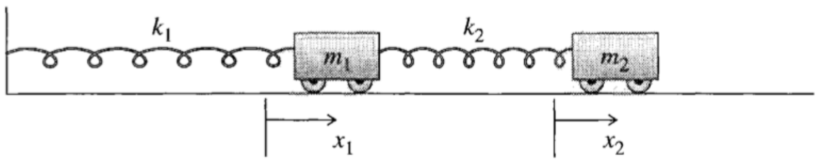
A mass m moves on a circular orbit centred on the origin in the field of an attractive central force with potential energy $U = kr^n$. State and prove the virial theorem! *Hint: You may use Newton's law in cylindrical coordinates (if you use anything else, you need to prove it first):*

$$\begin{aligned} F_r &= m\left(\ddot{r} - r\dot{\phi}^2\right) \\ F_\phi &= m\left(r\ddot{\phi} + 2\dot{r}\dot{\phi}\right). \end{aligned}$$

Question 5 **2 Points**

The Earth's orbit has a semi-major axis of 1 astronomical unit. The orbit of Venus has a semi-major axis of 0.723 astronomical units. For this question, you may assume that the planets' orbits are circular. You want to send a spacecraft from Earth to Venus using a Hohmann transfer orbit. You may launch the spacecraft at any time during the year. Calculate how long this journey will take! Your result should be accurate to three digits.

Question 6 **8 Points**



Calculate the Lagrangian for the system shown above!

Calculate the Lagrange equations and bring them into the standard form $\mathbf{K} \cdot \mathbf{x} = \mathbf{M} \cdot \ddot{\mathbf{x}}$!

Simplify the system by setting $k = k_1 = k_2$ and $m = m_1 = m_2$. Then calculate the frequencies of the normal modes!

We now change the system by keeping the distance between the two masses fixed with a massless rod. What is the corresponding constraint equation?

Calculate the Lagrange equations for the constrained system and solve them together with the constraint equation! What is the frequency of the normal mode in this case? Comment on whether your results is surprising to you!

Question 7

4 Points

Answer the following multiple choice questions! You get 1/2 point for every correct answer. But 1/2 point will be subtracted for every wrong answer. You can get a minimum of 0 points and a maximum of 4 points for this question.

(a) Classical mechanics only works on small length scales.

☐ true ☐ false

(b) Some concepts from classical mechanics such as symmetries and their connection to conservation laws are also applicable in Quantum Mechanics and general Relativity.

☐ true ☐ false

(c) A rocket burns all its fuel, corresponding to 90% of its initial mass. What is the maximum speed the rocket can gain in units of the exhaust speed v_{ex}?

☐ 1.7 ☐ 1.9 ☐ 2.1 ☐ 2.3

(d) The Work-KE theorem is only applicable to conservative forces.

☐ true ☐ false

(e) A spherically symmetric central force has which form?

☐ $\mathbf{F}(\mathbf{r}) = f(r)\,\hat{\mathbf{r}}$ ☐ $\mathbf{F}(\mathbf{r}) = f(\mathbf{r})\,\hat{\mathbf{r}}$ ☐ $\mathbf{F}(\mathbf{r}) = f(\hat{\mathbf{r}})\,r$ ☐ $\mathbf{F}(\mathbf{r}) = f(\hat{\mathbf{r}})\,\mathbf{r}$

(f) The FWHM is related to the HWHM in which way?

☐ $2 \cdot \mathrm{FWHM} = \mathrm{HWHM}$ ☐ $\mathrm{FWHM} = \mathrm{HWHM}$ ☐ $\mathrm{FWHM} = 2 \cdot \mathrm{HWHM}$

(g) The Lagrangian of any system has to be written in inertial coordinates.

☐ true ☐ false

(h) The Solar System is chaotic.

☐ true ☐ false

(i) The reduced mass μ in a two-body system is defined as how?

☐ $\frac{m_1 \cdot m_2}{m_1 - m_2}$ ☐ $\frac{m_1 \cdot m_2}{m_1 + m_2}$ ☐ $\frac{m_1 + m_2}{m_1 - m_2}$ ☐ $\frac{m_1 + m_2}{m_1 \cdot m_2}$

(j) Which of the following commands calculates the eccentricity of a particle in a `REBOUND` simulation?

☐ `sim.move_to_com()` ☐ `sim.calculate_e(1)` ☐ `sim.particles[1].e`
☐ `sim.e[1]` ☐ `sim.get_ecc()` ☐ `sim.particles[1].eccentricity`