Name: ____________________ Student number: ____________

Mechanics: From Oscillations to Chaos
PHYB54H3, Final Exam

Professor Hanno Rein
University of Toronto, Scarborough

Thursday, April 21st 2018, 2pm-4pm

	Points	Max Points (+Bonus Points)
Question 1		5
Question 2		3
Question 3		3
Question 4		3 (+2)
Question 5		5
Question 6		4 (+2)
Question 7		4 (+4)
Question 8		7 (+3)
Question 9		6
Question 10		7
Question 11		4 (+3)
Total		51 (+14)

- You may use a ruler and a non-programmable calculator. But no aid sheets, books or other notes are allowed.
- All electronic devices must be stored together with your belongings at the back of the room.
- Write your answers on the question sheet. If you need more paper raise your hand.
- The University of Toronto's Code of Behaviour on Academic Matters applies to all University of Toronto Scarborough students. The Code prohibits all forms of academic dishonesty including, but not limited to, cheating, plagiarism, and the use of unauthorized aids. Students violating the Code may be subject to penalties up to and including suspension or expulsion from the University.

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Question 1 **5 Points**

Check all systems which are *not* well described by classical mechanics!

- ☐ The motion of a raindrop falling to the ground.
- ☐ The chaotic motion of Jupiter trojan asteroid over long timescales.
- ☐ A particle travelling at almost (but not quite!) the speed of light in the Large Hadron Collider.
- ☐ Two cars colliding with each other at a relative speed of 150 km/h.
- ☐ The transition of electrons between different orbitals in the helium atom.
- ☐ The two merging black holes producing the famous GW150914 signal.

How do we define mass in classical mechanics?

- ☐ A particle's interaction with the Higgs field.
- ☐ By the amount of gravitational flux coming from the object.
- ☐ By measuring an object's weight.
- ☐ An object's resistance to acceleration.

Check all defining features that a point particle has!

- ☐ Vibrational energy
- ☐ Deformation constant
- ☐ Position
- ☐ Opacity
- ☐ Mass
- ☐ Rotational speed
- ☐ Natural frequency
- ☐ Velocity

Check all statements which apply to a conservative force F!

- ☐ The force only depends on the past (i.e. not the future).
- ☐ $\nabla \cdot F = 0$.
- ☐ The force only depends on the position.
- ☐ There exists no potential for the force.
- ☐ The work done by the force is path independent.

If a central force is conservative, then...

- ☐ it has the origin at the centre of mass.
- ☐ it has a linear potential.
- ☐ it is also spherically symmetric.
- ☐ it cannot be written down in closed form.

Question 2 **3 Points**

Quote Newton's first, second, and third law!

Question 3

3 Points

Describe how a rocket works! Focus on the aspects related to this course.

Question 4

3 Points ***(+ 2 Bonus Points)***

State and proof Kepler's second law! You may assume angular momentum is conserved.

Bonus part: Proof that the angular momentum is indeed conserved in this case.

Question 5 **5 Points**

Write down the total potential energy for a system consisting of N particles. Distinguish between the energies $U_{\alpha\beta}$ coming from internal pairwise interaction forces $F_{\alpha\beta}$, and the energies U_α coming from external forces F_α. Assume that internal and external forces are conservative. Pay particular attention to the summation indices.

Explain why Hooke's law, which says that the force needed to compress a spring by some distance x is linear with respect to the distance x, plays such an important role in many areas of physics.

Question 6

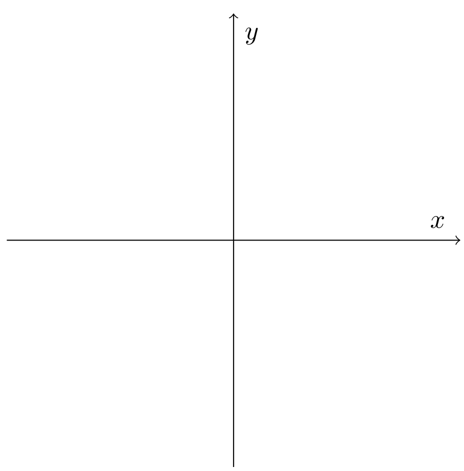
4 Points ***(+ 2 Bonus Points)***

$$\ddot{\vec{r}} = \begin{pmatrix} -4 & 0 \\ 0 & -16 \end{pmatrix} \cdot \vec{r} \qquad \text{where} \qquad \vec{r} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

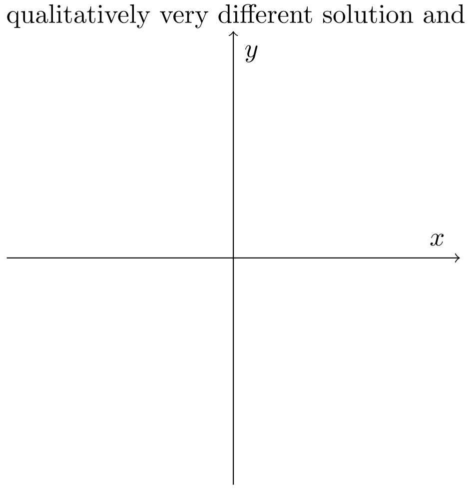
Write down the most general solution of the above set of differential equations in terms of sin and cos functions!

How many free parameters does your general solution have?

Sketch the solution in the xy-plane, assuming the system starts at the origin and the velocity points in the $(1, 1)$ direction!



Bonus part: For the same differential equation, choose different initial conditions which produce a qualitatively very different solution and sketch it.



Question 7

4 Points ***(+4 Bonus Points)***

Write down a second order linear homogeneous differential equation.

Write down a fourth order linear inhomogeneous differential equation.

Write down the differential equation that governs the evolution of the driven damped harmonic oscillator!

Characterise the differential equation! (What is the order of this differential equation? Is it linear or non-linear? Is it homogeneous or inhomogeneous?)

Bonus part: Assume the oscillator is initially at rest and we have a forcing term of the form $F(t) = F_0\,\delta(1)$ where δ is the Dirac delta function. Assume finite constants for all the other terms. Solve the differential equation in the weak damping limit! (Hint: We've never solved a differential equation with a delta function before. Think about what the solution looks like and then try to piece it together from what we discussed in the lectures.)

Question 8

7 Points *(+3 Bonus Points)*

Define the Lagrangian $\mathcal{L}$ for a particle in three dimensions under the influence of a conservative force F! Use Cartesian coordinates x, y, z.

Show that the Lagrange equations are just another way of writing Newton's second law!

Assume the particle's movement is restricted to a sphere of radius R around the origin. Write down this constraint as a constraint equation!

Write down all three modified Lagrange equations!

How many unknown functions and how many equations do you have in total?

Bonus part: Assume that $z(0) = \dot{z}(t) = 0$ and that the potential is given by $U(r) = x^2 + y^2 + z^2$. Then solve the system of four coupled differential equations (you may guess a solution)!

Question 9

6 Points

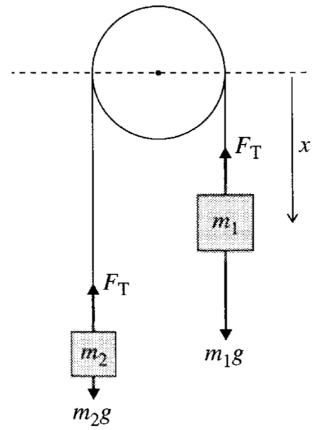
The picture on the right shows the Atwood machine. Calculate the equation of motion in three different ways:

(a) By applying Newton's second law to each mass.

(b) By differentiating the total energy and using the fact that the total energy is conserved.

(c) By using the Lagrange equation (you may either use generalized coordinates or the modified Lagrange equation).

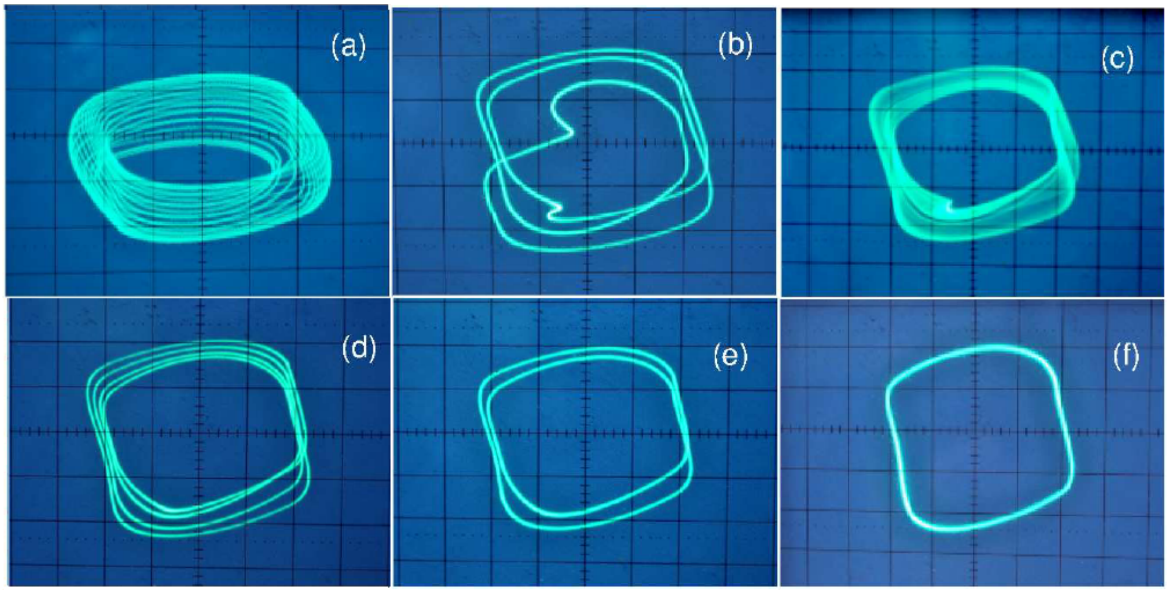
Express the equation of motion in terms of x and $\dot{x}$.



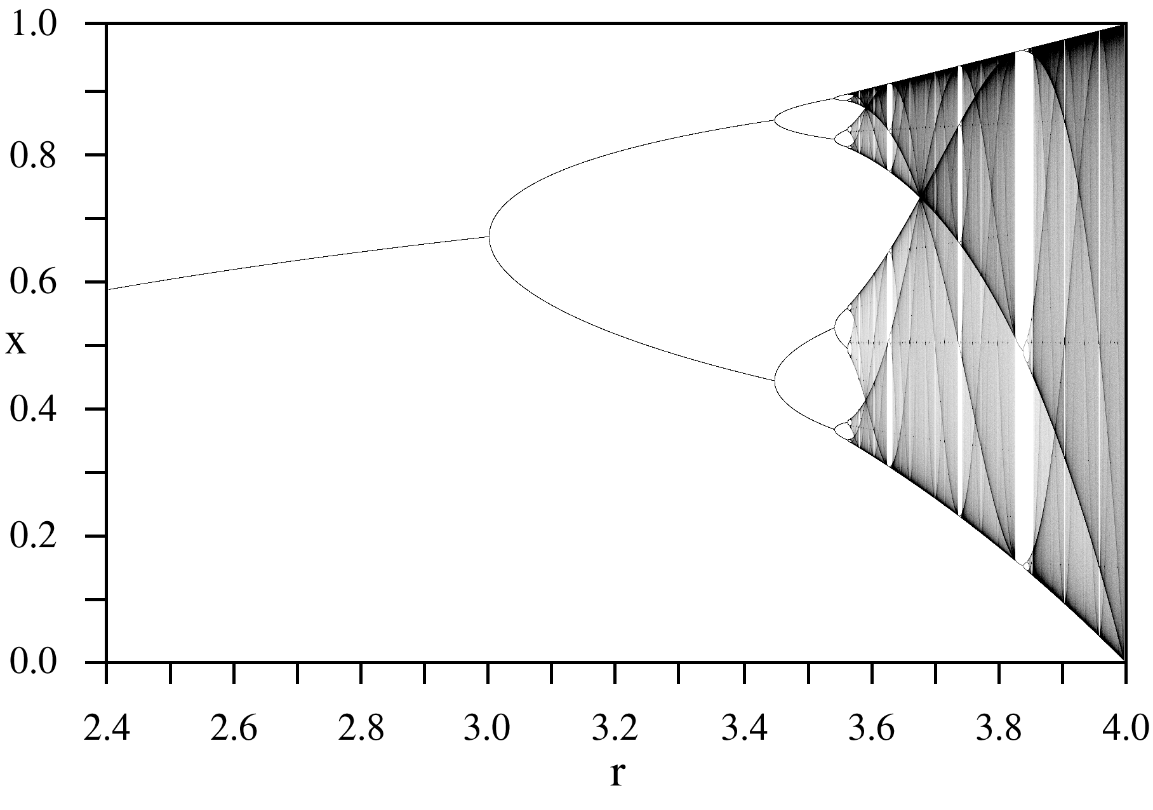
Question 10 **7 Points**

What is the LC circuit and how is it related to what we discussed in this course?

Is the LC circuit chaotic? If not, what needs to be added to allow for the existence of chaos?



The phase space images above were taken with a version of Chua's circuit. The Feigenbaum diagram below corresponds to the same circuit. Indicate with an arrow and the corresponding letter where in the Feigenbaum diagram each of the phase space images might occur.



Question 11

4 Points (+3 Bonus Points)

We have a dynamical system and perform an experiment with the initial conditions $\vec{\phi}_0$ and the slightly perturbed initial conditions $\vec{\phi}_0 + \epsilon$. The trajectories we observe are $\vec{\phi}(t)$ and $\vec{\phi}_\epsilon(t)$ respectively. Assume that the system is chaotic. How does the quantity $\Delta(t) \equiv |\vec{\phi}(t) - \vec{\phi}_\epsilon(t)|$ for some arbitrary norm $|\cdot|$ evolve in the limit of large t?

Is the behaviour you expect for $\Delta(t)$ unique to chaotic systems? If so why? If not, write down a differential equation that is clearly non-chaotic but shows the same behaviour!

Bonus Question: The Tesla Roadster launched by SpaceX earlier this year is currently in orbit around the sun. Is the orbit chaotic? Make use of your answer from above to discuss if it is possible to determine the long term evolution of the Roadster accurately!

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